Lecture 7. Properness & Valuation Criterion

Note Title 9/2/2019

Properness / Completeness is the analogue of Compactness for schemes

Definition: $f: X \to Y$ morphism of schemes

f is <u>proper</u> if it is separated, of finite type & <u>universally closed</u>

closed under base change

ex. $X = \mathbb{A}^1_k \to \operatorname{Spec} k$ separated, of finite type

$$\begin{array}{ccc} \mathbb{A}^1_k & \longrightarrow & \operatorname{Spec} k \\ \uparrow & & \uparrow \\ \mathbb{A}^1 \times_k \mathbb{A}^1_k & \longrightarrow & \mathbb{A}^1_k \\ \uparrow \text{closed} & & \uparrow \text{NOT closed} \\ \{xy=1\} & \longrightarrow & \mathbb{A}^1 \setminus \{0\} \end{array}$$

NOT universally closed

Theorem: (Valuation Criterion of Properness)

$f: X \to Y$ of finite type

f proper iff $\forall R$ valuation ring w/ quotient field K

$$\begin{array}{ccc} U = \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \overset{\exists !}{\nearrow} & \downarrow f \\ T = \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

Examples:

1. Closed immersions are proper
2. Composition of proper morphisms is proper
3. Proper morphisms stable under base change.
4. Product of proper morphisms is proper

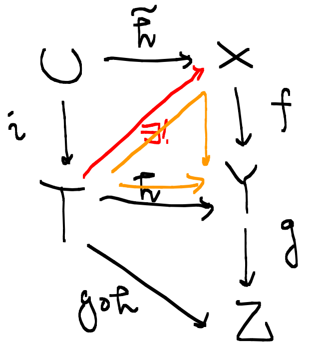
5. $X \xrightarrow{f} Y \xrightarrow{g} Z$. $g \circ f$ proper $\Longrightarrow$ f is proper
g seperated

pf: 5. • f is seperated? • Valuation Criterion?

$X \xrightarrow{f} Y \xrightarrow{g} Z$

f quasi-compact $\rightarrow$ f of finite type
$g \circ f$ of finite type

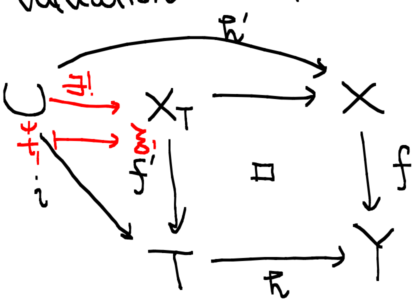
X: Notherian $\Rightarrow$ f quasi-compact
$g \circ f$ proper $\Rightarrow$ $g \circ f$ of finite type



g separated
$\Downarrow$
Compatibility

Proof of valuation criterion:

$(\Longrightarrow)$



$Z = \overline{\{\xi_1\}} \subseteq X_T$

f universally closed $\rightarrow$ $f'(Z) \subseteq T$ closed

i.e. $f'(Z) = T$

$\therefore$ $f(\xi_1) = t_1$, $f'(\xi_0) = t_0$
$\hat{Z}$

• $k(\xi_1) \hookrightarrow K$

$\mathcal{O} = \mathcal{O}_{\xi_0, Z}$ w/ quotient field $k(\xi_1)$

• $f' : Z \twoheadrightarrow T$ $\approx$ $R \xrightarrow{\cong} \mathcal{O}$

R is maximal local ring in K w.r.t domination

Lemma 3 $\Rightarrow$ $T \rightarrow X_T \rightarrow X$ (desire lifting $T \rightarrow X$)

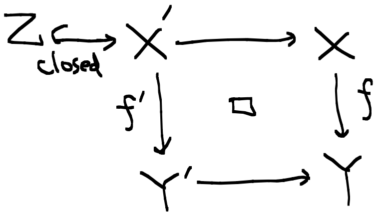
$f: X \rightarrow Y$ both integral
$\xi_X \mapsto \xi_Y$ generic point

then $\mathcal{O}_{\xi_Y} \hookrightarrow \mathcal{O}_{\xi_X}$

$K(\xi_Y) \hookrightarrow K(\xi_X)$

quotient field of $\mathcal{O}_{\xi_Y}$

$(\Longleftarrow)$ It suffices to show that f is universally closed



i.e $f'(Z)$ is closed in Y'

w/ Lemma 2

it suffices to prove that $f'(Z)$ is stable under specialization.

$Z \to Y'$ quasi-compact $\overset{\text{Lemma 4}}{\Longrightarrow}$

?

i.e $\left(z_1 \in Z,\ y_1 = f'(z_1) \quad y_0 \in \overline{\{y_1\}} \text{ then } y_0 \in f'(Z). \right)$

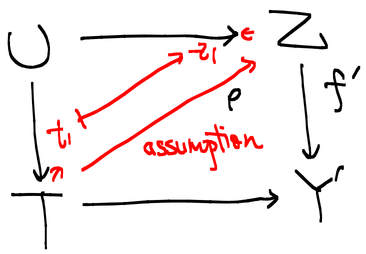
$\mathcal{O} = \mathcal{O}_{y_0, \overline{\{y_1\}}}$ local ring w/ quotient field $k(y_1)$

$\downarrow$

$k(z_1) = K$

Take $R =$ valuation ring of $k(z_1)$ dominating $\mathcal{O}$

$\rightsquigarrow$



$\overline{\{y_1\}} = \overline{\{f'(z_1)\}}$

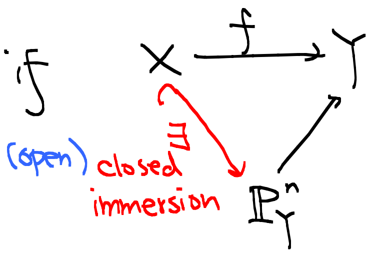
$= \overline{\{f'(\rho(t_1))\}}$

$\cup$

$f'(\rho(t_0)) = y_0$

Definition: (Projective space) A: ring. Z: scheme

- $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \cdots, x_n]$
- $\mathbb{P}^n_Z := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Z$
- $f: X \to Y$ is (quasi-) projective if



Theorem 2: A projective morphism between Noetherian schemes is proper.

pf: Examples 1.2 $\Longrightarrow$ only need to show $\mathbb{P}^n_Y \to Y$ proper

Example 3 $\Longrightarrow$ " $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ proper

$\mathbb{P}^n_{\mathbb{Z}} = \bigcup_i U_i$. $U_i = \operatorname{Spec}\mathbb{Z}\left[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right]$ of finite type

$$\begin{array}{ccc} t_1 \in U & \xrightarrow{\tilde{h}} & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & & \downarrow \\ t_0, t_1 \in T & \xrightarrow{h} & \operatorname{Spec}\mathbb{Z} \end{array}$$

$k(\xi_1) \hookrightarrow K$

WLOG may assume that $\tilde{h}(t_1) \in \bigcap U_i$ by induction

i.e $\frac{x_i}{x_j}$ invertible in $\mathcal{O}_{\tilde{h}(t_1)}$

$f_{ij} = \frac{x_i}{x_j} \in K$ w/ $f_{ij} \cdot f_{jk} = f_{ik} \in K$

$\mathrm{val}: K \longrightarrow G$
$f_{i0} \longmapsto g_i$

say g_k is minimal among $\{g_i\}$

then $\mathrm{val}(f_{ik}) = g_i - g_k \geq 0$ or $f_{ik} \in R$

$\rightsquigarrow \varphi: \mathbb{Z}\left[\frac{x_0}{x_k}, \cdots, \frac{x_n}{x_k}\right] \longrightarrow R \qquad \approx \operatorname{Spec} T \to U_k$

$\frac{x_i}{x_k} \longmapsto f_{ik}$

uniqueness $U_k \to \operatorname{Spec}\mathbb{Z}$ is separated
affine

Proposition: $t: \mathrm{Var}(k) \longrightarrow \mathrm{Sch}(k)$

$\searrow$ $\{$ quasi-projective integral schemes $/k \}$ $\nearrow$